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Unified Maxwell-Einstein and Yang-Mills-Einstein supergravity theories in five dimensions*

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Abstract: Unified $\mathcal{N}=2$ Maxwell-Einstein supergravity theories (MESGTs) are supergravity theories in which all the vector fields, including the graviphoton, transform in an irreducible representation of a simple global symmetry group of the lagrangian. As was established long time ago, in five dimensions there exist only four unified Maxwell-Einstein supergravity theories whose target manifolds are symmetric spaces. These theories are defined by the four simple euclidean Jordan algebras of degree three. In this paper, we show that, in addition to these four unified MESGTs with symmetric target spaces, there exist three infinite families of unified MESGTs as well as another exceptional one. These novel unified MESGTs are defined by non-compact (minkowskian) Jordan algebras, and their target spaces are in general neither symmetric nor homogeneous. The members of one of these three infinite families can be gauged in such a way as to obtain an infinite family of *unified* $\mathcal{N}=2$ Yang-Mills-Einstein supergravity theories, in which all vector fields transform in the adjoint representation of a simple gauge group of the type $\mathrm{SU}(N,1)$. The corresponding gaugings in the other two infinite families lead to Yang-Mills-Einstein supergravity theories coupled to tensor multiplets.

Keywords: Extra Large Dimensions, Gauge Symmetry, Global Symmetries, Supergravity Models.

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1. Introduction

One of the original motivations for studying supersymmetric theories in particle physics was the hope that they might provide the right framework for

a) unifying gravity with the gauge interactions of the Standard Model and

b) for putting the fermionic matter constituents on an equal footing with the bosonic fields that mediate the interactions between them.

In the early 1980s, efforts in this direction culminated in the construction of the maximally supersymmetric, $\mathcal{N} = 8$ supergravity theory with the gauge group SO(8) [1]. In this theory, all fields sit in one and the same supermultiplet and are thus connected by supersymmetry and/or gauge transformations.

It soon became clear, however, that neither the $\mathcal{N} = 8$ theory nor any other extended four-dimensional (4D) supergravity theory can be a phenomenologically realistic model of low energy particle physics, and that, instead, a realistic four-dimensional extension of the Standard Model can involve at most minimal $\mathcal{N} = 1$ supersymmetry. Moreover, gravity, the Yang-Mills gauge fields and the matter constituents all have to sit in different types of supermultiplets and can therefore not be connected by supersymmetry transformations or any other obvious symmetry. Thus, the original idea of using supersymmetry to directly unify all particles and interactions in terms of a purely 4D field theory did not prove to be successful.

Nevertheless, supersymmetry still plays an important rôle in the context of unification, albeit now in a more indirect way. For one thing, supersymmetry naturally appears in string theories, i.e., in the most promising known models for a complete unification of all particles and interactions, including gravity. Furthermore, from a more bottom-up point of view, a supersymmetrization of the Standard Model spectrum seems to be required in order to reconcile precision measurements at particle colliders with the idea of converging Standard Model couplings within a conventional GUT scenario [2].

Partly motivated by certain string theory constructions, such grand unified models have recently also been studied within a higher dimensional framework in order to overcome some of the notorious problems of standard 4D GUTs, such as proton decay or the doublet-triplet splitting problem. Especially five-dimensional models have been studied quite extensively in this context starting with refs. [3].

In five dimensions, the smallest possible amount of supersymmetry involves eight real supercharges, which, in analogy with the corresponding 4D terminology, is often referred to as $\mathcal{N}=2$ supersymmetry. Unlike its minimally (i.e., $\mathcal{N}=1$) supersymmetric counterpart in 4D, the 5D, $\mathcal{N}=2$ supergravity multiplet contains a vector field (the 'graviphoton'). It is therefore, in principle, conceivable to have an additional bosonic symmetry that could map the 5D graviphoton to some or all of the vector fields that sit in 5D vector multiplets. As supersymmetry interpolates between the graviphoton and the graviton, one might then, in a certain sense, view such a symmetry as a 'unification' of the vector multiplet sector and the gravity sector of the theory.

Such extra bosonic symmetries are of course nothing new, but constitute a well-known feature of extended supergravity theories, already in four dimensions. Consider for example 4D, $\mathcal{N}=4$ supergravity coupled to n abelian vector multiplets [4]. This theory contains $(6+n)$ vector fields, where 6 come from the supergravity multiplet (i.e., there are six 'graviphotons') and the remaining n are supplemented by the n vector multiplets. In addition to the local $\mathcal{N}=4$ supersymmetry, this theory has a global symmetry group of the form $G=\mathrm{SU}(1,1)\times\mathrm{SO}(6,n)$. Under the $\mathrm{SO}(6,n)$ factor, the $(6+n)$ vector fields of the theory transform irreducibly in the $(\mathbf{6}+\mathbf{n})$ representation, i.e., they are all connected by a symmetry of the theory even though they originate from different types of supermultiplets. In the following, we will call such an extended abelian supergravity theory in which *all* the vector fields transform *irreducibly* under a *simple* global symmetry group 'unified', or, more precisely, a 'unified' Maxwell-Einstein supergravity theory (unified MESGT). As the above example illustrates, the 'unifying' symmetry group of a unified MESGT is, in general, non-compact.

One might wonder whether it is also possible to have a similar 'unification' between graviphotons and vector fields from vector multiplets, when the latter gauge a non-abelian Yang-Mills symmetry. Let us first reconsider the above $\mathcal{N}=4$ theories. For the special case $n=3$, the global $\mathrm{SO}(6,3)$ symmetry has the obvious subgroup $\mathrm{SO}(2,1)\times\mathrm{SO}(2,1)\times\mathrm{SO}(2,1)$, under which the original $(\mathbf{6}+\mathbf{3})$ of $\mathrm{SO}(6,3)$ decomposes into three $\mathrm{SO}(2,1)$ triplets. Using standard supergravity techniques, one can then turn $\mathrm{SO}(2,1) \times \mathrm{SO}(2,1) \times \mathrm{SO}(2,1)$ into a Yang-Mills-type gauge symmetry under which all vector fields transform irreducibly in the adjoint representation. However, this gauge group is not *simple*, so that, in analogy to our abelian definition, we shall not call this theory 'unified'. Instead, we define a unified Yang-Mills-Einstein supergravity theory (unified YMESGT) to be a supergravity theory in which a *simple* Yang-Mills-type gauge group acts *irreducibly* on all graviphotons and all the vector fields that come from vector multiplets.

It is easy to convince oneself that such a theory cannot be constructed for our above 4D, $\mathcal{N} = 4$ examples, no matter how many vector multiplets are used. One might therefore wonder whether such unified YMESGTs exist at all.

As was first shown in ref. [5], the answer is in the affirmative. The example constructed in [5] (see also [6]) describes the coupling of 5D, $\mathcal{N} = 2$ supergravity to 14 vector multiplets and has the gauge group $\mathrm{SU}(3,1)$. This gauge group is possible because the graviphoton of the $\mathcal{N} = 2$ supergravity multiplet and the 14 vector fields of the 14 vector multiplets combine into the 15-dimensional adjoint representation of $\mathrm{SU}(3,1)$. Turning off the Yang-Mills coupling, one obtains a unified MESGT in which the global unifying group gets enhanced to $\mathrm{SU}^*(6)$. In [7], this unified MESGT with 14 vector multiplets was found to be a member of a family of four unified MESGTs, which have, respectively, 5, 8, 14 or 26 vector multiplets and are associated with the simple Jordan algebras of hermitean (3×3)-matrices over the four division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. If one restricts oneself to theories in which the scalar manifold forms a symmetric space, these four theories are the only unified MESGTs in 5D. Except for the theory with 14 vector multiplets, none of them can be turned into a unified YMESGT upon gauging a subgroup of the relevant global symmetry groups. In the class of theories with symmetric target spaces, this theory is thus unique.

In this paper we show that if one abandons the restriction to *symmetric* target spaces, there are three infinite families of novel unified MESGTs in 5D, $\mathcal{N} = 2$ supergravity as well as one novel exceptional unified MESGT. One of the three infinite families can even be turned into an infinite family of unified YMESGTs with gauge groups of the type $\mathrm{SU}(1,\mathrm{N})$ for arbitrary high $N \geq 2$. We can show that these theories exhaust all possible unified YMESGTs in five dimensions (the theory found in [5, 6] turns out to be a special case of this infinite family). As a by-product, we find an intriguing connection to a classical work by Elie Cartan on some remarkable families of isoparametric hypersurfaces in spaces of constant curvature [8].

The paper is organized as follows. In section 2, we briefly recall the basic properties of $\mathcal{N} = 2$ MESGTs in five dimensions. Section 3 gives a short description of the four unified MESGTs with symmetric target spaces that were found in ref. [7]. In section 4 we recall the relation of these four theories to Jordan algebras and show how the language of Jordan algebras quickly leads to the construction of three novel infinite families as well as another exceptional unified MESGT. The members of only one of the three infinite families can be turned into unified YMESGTs, which is further explained in section 5. In the course of this work, we became aware of a connection to earlier work by E. Cartan on isoparametric hypersurfaces in spaces of constant curvature [8]. This connection is sketched in section 6. Our results are summarized and discussed in section 7.

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2. 5D, $\mathcal{N}=2$ Maxwell-Einstein supergravity theories

In this section, we review the salient features of general 5D, $\mathcal{N}=2$ Maxwell-Einstein supergravity theories (MESGTs) [7].[1]

A 5D, $\mathcal{N}=2$ MESGT describes the coupling of pure 5D, $\mathcal{N}=2$ supergravity to an arbitrary number, $\tilde{n}$, of vector multiplets. The fields of the supergravity multiplet are the fünfbein e_μ^m, two gravitini Ψ_μ^i ($i=1,2$) and one vector field A_μ (the graviphoton). An $\mathcal{N}=2$ vector multiplet contains a vector field A_μ, two spin-1/2 fermions λ^i and one real scalar field φ. The fermions of each of these multiplets transform as doublets under the $\mathrm{USp}(2)_R \cong \mathrm{SU}(2)_R$ R-symmetry group of the $\mathcal{N}=2$ Poincaré superalgebra; all other fields are $\mathrm{SU}(2)_R$-inert.

Putting everything together, the total field content of an $\mathcal{N}=2$ MESGT is thus

$$\{e_\mu^m, \Psi_\mu^i, A_\mu^{\tilde{I}}, \lambda^{i\tilde{a}}, \varphi^{\tilde{x}}\} \tag{2.1}$$

with

$$\begin{aligned} \tilde{I} &= 0,1,\ldots,\tilde{n} \\ \tilde{a} &= 1,\ldots,\tilde{n} \\ \tilde{x} &= 1,\ldots,\tilde{n}\,. \end{aligned}$$

Here, we have combined the graviphoton with the $\tilde{n}$ vector fields of the $\tilde{n}$ vector multiplets into a single $(\tilde{n}+1)$-plet of vector fields $A_\mu^{\tilde{I}}$ labelled by the index $\tilde{I}$. The indices $\tilde{a},\tilde{b},\ldots$ and $\tilde{x},\tilde{y},\ldots$ denote the flat and curved indices, respectively, of the $\tilde{n}$-dimensional target manifold, $\mathcal{M}$, of the scalar fields.

The bosonic part of the lagrangian is given by (for the fermionic part and further details see [7])

$$e^{-1}\mathcal{L}_{\text{bosonic}} = -\frac{1}{2}R - \frac{1}{4}\overset{\circ}{a}_{\tilde{I}\tilde{J}} F_{\mu\nu}^{\tilde{I}} F^{\tilde{J}\mu\nu} - \frac{1}{2} g_{\tilde{x}\tilde{y}}(\partial_\mu\varphi^{\tilde{x}})(\partial^\mu\varphi^{\tilde{y}}) + \frac{e^{-1}}{6\sqrt{6}} C_{\tilde{I}\tilde{J}\tilde{K}} \varepsilon^{\mu\nu\rho\sigma\lambda} F_{\mu\nu}^{\tilde{I}} F_{\rho\sigma}^{\tilde{J}} A_\lambda^{\tilde{K}}\,, \tag{2.2}$$

where e and R denote the fünfbein determinant and the scalar curvature, respectively, and $F_{\mu\nu}^{\tilde{I}}$ are the abelian field strengths of the vector fields $A_\mu^{\tilde{I}}$. The metric, $g_{\tilde{x}\tilde{y}}$, of the scalar manifold $\mathcal{M}$ and the matrix $\overset{\circ}{a}_{\tilde{I}\tilde{J}}$ both depend on the scalar fields $\varphi^{\tilde{x}}$. The completely symmetric tensor $C_{\tilde{I}\tilde{J}\tilde{K}}$, by contrast, is constant. Remarkably, the entire $\mathcal{N}=2$ MESGT (including also the fermionic terms and the supersymmetry transformation laws we have not shown here) is uniquely determined by the $C_{\tilde{I}\tilde{J}\tilde{K}}$ [7]. More explicitly, the $C_{\tilde{I}\tilde{J}\tilde{K}}$ define a cubic polynomial, $\mathcal{V}(h)$, in $(\tilde{n}+1)$ real variables $h^{\tilde{I}}$ ($\tilde{I}=0,1,\ldots,\tilde{n}$),

$$\mathcal{V}(h) := C_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{I}} h^{\tilde{J}} h^{\tilde{K}}\,. \tag{2.3}$$

This polynomial defines a metric, $a_{\tilde{I}\tilde{J}}$, in the (auxiliary) space $\mathbb{R}^{(\tilde{n}+1)}$ spanned by the $h^{\tilde{I}}$:

$$a_{\tilde{I}\tilde{J}}(h) := -\frac{1}{3}\frac{\partial}{\partial h^{\tilde{I}}}\frac{\partial}{\partial h^{\tilde{J}}} \ln\mathcal{V}(h)\,. \tag{2.4}$$

[1] Our conventions coincide with those of ref. [7, 6, 9]. In particular, we will use the mostly positive metric signature $(-++++)$ and impose the 'symplectic' Majorana condition on all fermionic quantities.

The $\tilde{n}$-dimensional target space, $\mathcal{M}$, of the scalar fields $\varphi^{\tilde{x}}$ can then be represented as the hypersurface [7]

$$\mathcal{V}(h) = C_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{I}} h^{\tilde{J}} h^{\tilde{K}} = 1 \,, \tag{2.5}$$

with $g_{\tilde{x}\tilde{y}}$ being the pull-back of (2.4) to $\mathcal{M}$. The quantity $\overset{\circ}{a}_{\tilde{I}\tilde{J}}(\varphi)$ appearing in (2.2), finally, is given by the componentwise restriction of $a_{\tilde{I}\tilde{J}}$ to $\mathcal{M}$:

$$\overset{\circ}{a}_{\tilde{I}\tilde{J}}(\varphi) = a_{\tilde{I}\tilde{J}}|_{\mathcal{V}=1} \,.$$

The physical requirement of unitarity requires $g_{\tilde{x}\tilde{y}}$ and $\overset{\circ}{a}_{\tilde{I}\tilde{J}}$ to be positive definite. This requirement induces constraints on the possible $C_{\tilde{I}\tilde{J}\tilde{K}}$, and in [7] it was shown that any $C_{\tilde{I}\tilde{J}\tilde{K}}$ that satisfy these constraints can be brought to the following form

$$C_{000} = 1 \,, \qquad C_{0ij} = -\frac{1}{2}\delta_{ij} \,, \qquad C_{00i} = 0 \,, \tag{2.6}$$

with the remaining coefficients C_{ijk} $(i, j, k = 1, 2, \ldots, \tilde{n})$ being completely arbitrary. We shall refer to this basis as the canonical basis. The arbitrariness of the C_{ijk} in the canonical basis implies that, for a fixed number $\tilde{n}$ of vector multiplets, different target manifolds $\mathcal{M}$ are, in general, possible.

3. Unified MESGTs with symmetric target spaces

In this paper, we are interested in 'unified' Maxwell-Einstein supergravity theories, in which a simple global symmetry group acts irreducibly on *all* the vector fields $A_\mu^{\tilde{I}}$.

In general, the global symmetries of a 5D, $\mathcal{N} = 2$ MESGT can be divided into two categories:

- Any $\mathcal{N} = 2$ MESGT is always invariant under the global R-symmetry group $\mathrm{SU}(2)_R$. As mentioned earlier, $\mathrm{SU}(2)_R$ acts nontrivially only on the fermions Ψ_μ^i and $\lambda^{i\tilde{a}}$. In particular, it does not act on the vector fields $A_\mu^{\tilde{I}}$.
- Any group, G, of linear transformations

$$h^{\tilde{I}} \to B^{\tilde{I}}{}_{\tilde{J}} h^{\tilde{J}} \,, \qquad A_\mu^{\tilde{I}} \to B^{\tilde{I}}{}_{\tilde{J}} A_\mu^{\tilde{J}} \tag{3.1}$$

that leave the tensor $C_{\tilde{I}\tilde{J}\tilde{K}}$ invariant

$$B^{\tilde{I}'}{}_{\tilde{I}} B^{\tilde{J}'}{}_{\tilde{J}} B^{\tilde{K}'}{}_{\tilde{K}} C_{\tilde{I}'\tilde{J}'\tilde{K}'} = C_{\tilde{I}\tilde{J}\tilde{K}} \,,$$

is automatically a symmetry of the entire lagrangian (2.2), since the latter is uniquely determined by the $C_{\tilde{I}\tilde{J}\tilde{K}}$. These symmetries act as isometries of the scalar manifold $\mathcal{M}$, which becomes evident if one rewrites the kinetic energy term for the scalar fields as [7, 10]

$$-\frac{1}{2} g_{\tilde{x}\tilde{y}} (\partial_\mu \varphi^{\tilde{x}})(\partial^\mu \varphi^{\tilde{y}}) = \frac{3}{2} C_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{I}} \partial_\mu h^{\tilde{J}} \partial^\mu h^{\tilde{K}} \,,$$

with the $h^{\tilde{I}}$ being constrained according to (2.5).

As there is no interference between these two types of symmetries (a consequence of the vector and scalar fields being $\mathrm{SU}(2)_R$-inert), the full global symmetry group of (2.2) factorizes into $\mathrm{SU}(2)_R \times G$. Obviously, then, any 'unifying' symmetry group has to be a subgroup of G, because only this group acts non-trivially on the vector fields.

For a generic MESGT, the invariance group G of the underlying cubic polynomial $\mathcal{V}(h)$ can be rather small or even trivial. A well-studied class of theories with rather large symmetry groups G are the ones whose target spaces $\mathcal{M}$ are symmetric spaces. This class of MESGTs can be divided into three families [7, 11]:

1. The "generic" or "reducible" Jordan family:
$$\mathcal{M} = \frac{\mathrm{SO}(\tilde{n}-1,1)}{\mathrm{SO}(\tilde{n}-1)} \times \mathrm{SO}(1,1)\,, \qquad \tilde{n} \geq 1\,. \tag{3.2}$$

2. The "irreducible" or "magical" Jordan family.[2]
$$\begin{aligned} \mathcal{M} &= \mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(3) && (\tilde{n} = 5)\\ \mathcal{M} &= \mathrm{SL}(3,\mathbb{C})/\,\mathrm{SU}(3) && (\tilde{n} = 8)\\ \mathcal{M} &= \mathrm{SU}^*(6)/\,\mathrm{USp}(6) && (\tilde{n} = 14)\\ \mathcal{M} &= E_{6(-26)}/F_4 && (\tilde{n} = 26) \end{aligned}$$

3. The symmetric non-Jordan family:
$$\mathcal{M} = \frac{\mathrm{SO}(1,\tilde{n})}{\mathrm{SO}(\tilde{n})}\,, \qquad \tilde{n} > 1\,. \tag{3.3}$$

The reason for the names of these three families will become clear in the next section. Here, we focus on the question which of the above theories are unified MESGTs. Let us first consider the generic Jordan family (i). For this family, the isometry group of the scalar manifold $\mathcal{M}$ is given by $\mathrm{SO}(\tilde{n}-1,1) \times \mathrm{SO}(1,1)$. This is also the the symmetry group, G, of the underlying cubic polynomial
$$\mathcal{V}(h) = \frac{3\sqrt{3}}{2} h^0[(h^1)^2 - (h^2)^2 - \cdots - (h^{\tilde{n}})^2]\,, \tag{3.4}$$
where $\mathrm{SO}(1,1)$ acts by rescalings $(h^0, h^1, \ldots, h^{\tilde{n}}) \rightarrow (\lambda^2 h^0, \lambda^{-1}h^1, \ldots, \lambda^{-1}h^{\tilde{n}})$, and $(h^1, \ldots, h^{\tilde{n}})$ transform in the fundamental representation of $\mathrm{SO}(\tilde{n}-1,1)$. As h^0 is inert under $\mathrm{SO}(\tilde{n}-1,1)$, there can be no simple subgroup of G under which all $h^{\tilde{I}}$ (and thus all vector fields $A_\mu^{\tilde{I}}$) transform irreducibly. Hence, the generic Jordan family does not contain any unified MESGTs.

Let us now turn to the "magical" Jordan family (ii). For these theories, the isometry groups of the scalar manifolds $\mathcal{M}$ are $\mathrm{SL}(3,\mathbb{R})$, $\mathrm{SL}(3,\mathbb{C})$, $\mathrm{SU}^*(6)$ and $E_{6(-26)}$, respectively. Just as in the generic Jordan family (i), these are also the symmetry groups G of the underlying cubic polynomials $\mathcal{V}(h)$. Under these simple symmetry groups G, the, respectively, 6, 9, 15 and 27 vector fields $A_\mu^{\tilde{I}}$ transform irreducibly [7]. Thus, according to our definition, all four theories of the magical Jordan family are unified MESGTs.

[2]The name "magical" derives from the deep connection with the "magic square" of Freudenthal, Rosenfeld and Tits [12].

For the two families we have discussed so far, the symmetry group G of the $C_{\tilde{I}\tilde{J}\tilde{K}}$ always coincided with the full isometry group of the scalar manifold $\mathcal{M}$. For the third family (iii) of symmetric spaces, i.e. for the symmetric non-Jordan family, this is no longer true [13]: Whereas the isometry group of $\mathcal{M}$ is $\mathrm{SO}(1,\tilde{n})$, the symmetry group G of the $C_{\tilde{I}\tilde{J}\tilde{K}}$ (i.e., the symmetry group of the whole lagrangian) is only the subgroup $E_{(\tilde{n}-1)} \times \mathrm{SO}(1,1)$, where $E_{(\tilde{n}-1)}$ denotes the euclidean group in $(\tilde{n}-1)$ dimensions,

$$E_{(\tilde{n}-1)} = \mathrm{SO}(\tilde{n}-1) \ltimes T_{(\tilde{n}-1)} \,,$$

where $T_{(\tilde{n}-1)}$ is the group of translations in an $(\tilde{n}-1)$ dimensional euclidean space [13], and $\ltimes$ denotes the semi-direct product. A simple subgroup of this group has to be a subgroup of $\mathrm{SO}(\tilde{n}-1)$, under which only $(\tilde{n}-1)$ of the $(\tilde{n}+1)$ vector fields transform nontrivially [13]. Hence, the symmetric non-Jordan family does not provide us with any new unified MESGTs.

To sum up, of all the 5D, $\mathcal{N}=2$ MESGTs whose scalar manifolds are symmetric spaces only those of the magical Jordan family (ii) are *unified* MESGTs. This statement can even be extended to the larger class of theories in which $\mathcal{M}$ is *homogeneous*, but not necessarily symmetric. The possible homogeneous scalar manifolds were classified in [10], and it is easy to see from the list given in [10] that also in that class the only possible unified MESGTs are provided by the four magical theories described above.

The goal of the first part of this paper is to find more examples of unified MESGTs by abandoning the restriction that $\mathcal{M}$ be a symmetric or a homogeneous space.[3] The mathematical problem one has to solve in order to find such theories is easily stated: Find irreducible representations of simple groups G with an invariant third rank symmetric tensor $C_{\tilde{I}\tilde{J}\tilde{K}}$ such that the resulting metrics $g_{\tilde{x}\tilde{y}}$ and $\overset{\circ}{a}_{\tilde{I}\tilde{J}}$ are positive definite, at least in the vicinity of some point, c, on the hypersurface $C_{\tilde{I}\tilde{J}\tilde{K}}h^{\tilde{I}}h^{\tilde{J}}h^{\tilde{K}}=1$. An equivalent way of stating the positivity properties of $g_{\tilde{x}\tilde{y}}$ and $\overset{\circ}{a}_{\tilde{I}\tilde{J}}$ is to require that the $C_{\tilde{I}\tilde{J}\tilde{K}}$ can be brought to the canonical form (2.6) by means of a linear redefinition of the $h^{\tilde{J}}$.

In the following section, we show that an infinite number of solutions to this mathematical problem can be easily constructed using the language of Jordan algebras.

4. Jordan algebras and unified MESGTs

As we have seen in the previous section, within the class of symmetric or homogeneous scalar manifolds only four give rise to a unified MESGT. These four theories are all members of what we called the "magical Jordan family" (family (ii)). The magical Jordan family and the generic Jordan family (family (i)) owe their names to the fact that they are associated with Jordan algebras. The "symmetric non-Jordan family" (family (iii)), by contrast, is,

[3]In fact, there is no good physical motivation for such a restriction. For one thing, the quantum corrected low energy effective actions of $\mathcal{N}=2$ compactifications of string or M-theory are in general not based on homogeneous target spaces (see e.g. [14, 15] for some explicit 5D examples). Moreover, even in an intrinsically 5D framework, the deviation from the class of homogeneous target spaces is often a crucial step towards obtaining models with interesting physical properties (see e.g. [16, 17, 18]).

as the name suggests, not connected to Jordan algebras. In order to become more explicit, let us first recall the definition of a Jordan algebra.

Definition 1. *A Jordan algebra over a field $\mathbb{F}$ (which we take to be $\mathbb{R}$ or $\mathbb{C}$) is an algebra, J, over $\mathbb{F}$ with a symmetric product $\circ$,*

$$X \circ Y = Y \circ X \in J, \quad \forall\, X, Y \in J \;, \tag{4.1}$$

that satisfies the Jordan identity

$$X \circ (Y \circ X^2) = (X \circ Y) \circ X^2 \;, \tag{4.2}$$

where $X^2 \equiv (X \circ X)$.

The Jordan identity (4.2) is automatically satisfied when the product $\circ$ is associative, but (4.2) does not imply associativity. In other words, a Jordan algebra is commutative, but in general not associative. Historically, Jordan algebras were introduced in an attempt to generalize the formalism of quantum mechanics by capturing the algebraic essence of hermitean operators corresponding to observables without reference to the underlying Hilbert space on which they act [19]. While the hermitean operators acting on a Hilbert space do not close under the ordinary (associative) operator product, they do close and form a Jordan algebra under the Jordan product $\circ$ defined as one half the anticommutator.[4]

For every Jordan algebra J, one can define a norm form, $N : J \to \mathbb{R}$, that satisfies the composition property [20]

$$N[2X \circ (Y \circ X) - (X \circ X) \circ Y] = N^2(X) N(Y)\,. \tag{4.3}$$

The degree, p, of the norm form is defined by $N(\lambda X) = \lambda^p N(X)$, where $\lambda \in \mathbb{R}$. p is also called the degree of the Jordan algebra.

Definition 2. *A* euclidean *Jordan algebra is a Jordan algebra for which the condition $X \circ X + Y \circ Y = 0$ implies that $X = Y = 0$ for all $X, Y \in J$. The automorphism groups of euclidean Jordan algebras are always compact.*

In [7], it was shown, that whenever a Jordan algebra is euclidean, and its norm form N is cubic ($p = 3$), one can identify the norm form N with the cubic polynomial $\mathcal{V}$ of a MESGT so that $g_{\tilde{x}\tilde{y}}$ and $\overset{\circ}{a}_{\tilde{I}\tilde{J}}$ are positive definite. The MESGTs whose cubic polynomial arise in this way, are precisely the first two families (i) and (ii). The relevant Jordan algebras are

1. $J = \mathbb{R} \oplus \Sigma_{\tilde{n}}$ for the generic Jordan family with the scalar manifolds

   $$\mathcal{M} = \frac{\mathrm{SO}(\tilde{n}-1, 1)}{\mathrm{SO}(\tilde{n}-1)} \times \mathrm{SO}(1,1)\,.$$

   Here, $\Sigma_{\tilde{n}}$ is a Jordan algebra of degree $p = 2$ associated with a quadratic norm form in $\tilde{n}$ dimensions that has a "minkowskian signature" $(+, -, \ldots, -)$. A simple realization

[4] The axioms (4.1) and (4.2) are of course also fulfilled for other multiples of the anticommutator. The prefactor one half is singled out as the only prefactor for which eq. (4.3) becomes independent of the degree p of the Jordan algebra.

of $\Sigma_{\tilde{n}}$ is provided by $(\tilde{n}-1)$ Dirac gamma matrices γ^i $(i,j,\ldots = 1,\ldots,(\tilde{n}-1))$ of an $(\tilde{n}-1)$ dimensional euclidean space together with the identity matrix $\gamma^0 = \mathbf{1}$ and the Jordan product $\circ$ being one half the anticommutator:

$$\begin{aligned} \gamma^i \circ \gamma^j &= \frac{1}{2}\{\gamma^i, \gamma^j\} = \delta^{ij}\mathbf{1} \\ \gamma^0 \circ \gamma^0 &= \frac{1}{2}\{\gamma^0, \gamma^0\} = \mathbf{1} \\ \gamma^i \circ \gamma^0 &= \frac{1}{2}\{\gamma^i, \gamma^0\} = \gamma^i \,. \end{aligned} \tag{4.4}$$

The norm of a general element $X = X_0\gamma^0 + X_i\gamma^i$ of $\Sigma_{\tilde{n}}$ is defined as

$$N(X) = \frac{1}{2^{[\tilde{n}/2]}}\mathrm{Tr}X\bar{X} = X_0X_0 - X_iX_i\,,$$

where

$$\bar{X} \equiv X_0\gamma^0 - X_i\gamma^i\,.$$

The norm of a general element $y \oplus X$ of the non-simple Jordan algebra $J = \mathbb{R} \oplus \Sigma_{\tilde{n}}$ is simply given by $yN(X)$ (cf eq. (3.4)).

2. The magical Jordan family corresponds to the four *simple* euclidean Jordan algebras of degree 3. These simple Jordan algebras are denoted by $J_3^{\mathbb{R}}$, $J_3^{\mathbb{C}}$, $J_3^{\mathbb{H}}$, $J_3^{\mathbb{O}}$ and are isomorphic to the hermitean (3×3)-matrices over the four division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ with the product being one half the anticommutator:

$$\begin{array}{lll} J_3^{\mathbb{R}}: & \mathcal{M} = \mathrm{SL}(3,\mathbb{R})/\,\mathrm{SO}(3) & (\tilde{n} = 5) \\ J_3^{\mathbb{C}}: & \mathcal{M} = \mathrm{SL}(3,\mathbb{C})/\,\mathrm{SU}(3) & (\tilde{n} = 8) \\ J_3^{\mathbb{H}}: & \mathcal{M} = \mathrm{SU}^*(6)/\,\mathrm{USp}(6) & (\tilde{n} = 14) \\ J_3^{\mathbb{O}}: & \mathcal{M} = E_{6(-26)}/F_4 & (\tilde{n} = 26)\,. \end{array} \tag{4.5}$$

The cubic norm form, N, of these Jordan algebras is given by the determinant of the corresponding hermitean (3×3)-matrices.

In all the above examples, the scalar manifold is given by $\mathcal{M} = \frac{\mathrm{Str}_0(J)}{\mathrm{Aut}(J)}$, where $\mathrm{Str}_0(J)$ and $\mathrm{Aut}(J)$ are, respectively, the reduced structure group[5] and the automorphism group of the corresponding Jordan algebra J [7, 21].

4.1 The novel families of unified MESGTs defined by simple non-compact Jordan algebras

In this section we shall investigate the question whether there exist unified $\mathcal{N} = 2$ MESGTs beyond the four magical ones. We find that there do indeed exist three novel infinite

[5]The reduced structure group $\mathrm{Str}_0(J)$ is simply the invariance group of the norm form, N, of the corresponding Jordan algebra J. As such, it is, for the above Jordan algebras, isomorphic to the symmetry group G of the corresponding MESGTs.

families of unified $\mathcal{N}=2$ MESGTs, as well as another exceptional unified MESGT beyond the magical theories. The three infinite families are associated with Jordan algebras of arbitrary high degree $p \geq 3$ that are no longer euclidean.

Let us first try to understand heuristically why Jordan algebras also play a natural rôle in these novel families of unified MESGTs. In the two Jordan families (i) and (ii) discussed in the previous section, the cubic polynomial $\mathcal{V}$ defined by the symmetric tensor $C_{\tilde{I}\tilde{J}\tilde{K}}$ of the supergravity theory is identified with the norm form, N, of a euclidean Jordan algebra of degree three. As a consequence, the invariance group of the norm form becomes a symmetry group, G, of the supergravity lagrangian. Clearly, the restriction to Jordan algebras of degree three is crucial for this identification, because for Jordan algebras of degree $p > 3$ the norm forms are, by definition, no longer cubic. So, if we are to find new unified MESGTs, we will certainly not find them by identifying the cubic polynomial $\mathcal{V}$ with norm forms of other Jordan algebras. All the cases where a norm form of a Jordan algebra can be identified with an admissible supergravity polynomial $\mathcal{V}$ are already exhausted by the two families we discussed in the previous section.

In order to find new unified MESGTs, one would therefore have to identify the $C_{\tilde{I}\tilde{J}\tilde{K}}$ of such a MESGT with another mathematical object that admits the action of a non-trivial invariance group G. As a rather natural object of this sort, one could try the structure constants of an algebra. The fact that the $C_{\tilde{I}\tilde{J}\tilde{K}}$ are completely symmetric in three indices implies that if they are to be identified with the structure constants of some algebra, that algebra must have a symmetric product. Jordan algebras are, of course, some of the best known and studied algebras with a symmetric product. Furthermore, Jordan algebras are the natural mathematical structures that arise in the study of domains of positivity [22], which in our case are related to the positivity of the kinetic energy terms of the scalar and vector fields.

We are thus led to investigate the possibility of identifying the *structure constants* of Jordan algebras with the constants $C_{\tilde{I}\tilde{J}\tilde{K}}$ of the supergravity theory. However, if we are to identify the $C_{\tilde{I}\tilde{J}\tilde{K}}$ with the full set of structure constants of a simple Jordan algebra then the corresponding $\mathcal{N}=2$ MESGT can not be a *unified* theory since the invariance group of the structure constants is the automorphism group, under which the identity element of a simple Jordan algebra is a singlet. Because of this singlet, the automorphism group would then not act irreducibly on all the vector fields of the theory. Therefore, to be able to obtain a *unified* MESGT, we can at best try to identify $C_{\tilde{I}\tilde{J}\tilde{K}}$ with a subset of the structure constants that does not involve the identity element.

Furthermore, from the general form of the constants $C_{\tilde{I}\tilde{J}\tilde{K}}$ in the canonical basis we expect any symmetry group under which all the vectors transform in a single irreducible representation to be non-compact [7, 13, 17]. Thus we are led to investigate Jordan algebras with non-compact automorphism groups.

The Jordan algebras of $(n \times n)$ hermitean matrices over various division algebras are euclidean (compact) Jordan algebras whose automorphism groups are compact. In particular, the Jordan algebras of degree 3 discussed in the previous section are all euclidean. Non-compact analogs $J^{\mathbb{A}}_{(q,n-q)}$ of euclidean Jordan algebras $J^{\mathbb{A}}_n$ of $(n \times n)$ hermitean ma-

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trices over the associative division algebras $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ for $n \geq 3$ and of $J_3^{\mathbb{O}}$ are realized[6] by matrices that are hermitean with respect to a non-euclidean "metric" η with signature $(q, n-q)$:

$$(\eta X)^\dagger = \eta X \qquad \forall X \in J^{\mathbb{A}}_{(q,n-q)} \, . \tag{4.6}$$

Obviously, if we choose η to have a euclidean signature, we obtain back euclidean (compact) Jordan algebras. Consider now minkowskian Jordan algebras $J^{\mathbb{A}}_{(1,\mathrm{N})}$ of degree $n = N+1$ defined by choosing η to be the Minkowski metric $\eta = (-, +, +, \ldots, +)$. A general element, U, of $J^{\mathbb{A}}_{(1,\mathrm{N})}$ can be written in the form

$$U = \begin{pmatrix} x & -Y^\dagger \\ Y & Z \end{pmatrix} , \tag{4.7}$$

where Z is an element of the euclidean subalgebra $J^{\mathbb{A}}_N$ (i.e., it is a hermitean $(N \times N)$-matrix over $\mathbb{A}$), $x \in \mathbb{R}$, and Y denotes an N-dimensional column vector over $\mathbb{A}$. Under the automorphism group, $\mathrm{Aut}(J^{\mathbb{A}}_{(1,\mathrm{N})})$, the simple Jordan algebras $J^{\mathbb{A}}_{(1,\mathrm{N})}$ decompose into an irreducible representation formed by the traceless elements plus a singlet, which is given by the identity element of $J^{\mathbb{A}}_{(1,\mathrm{N})}$ (i.e., by the unit matrix $U = \mathbf{1}$):

$$J^{\mathbb{A}}_{(1,\mathrm{N})} = \mathbf{1} \oplus \{\text{traceless elements}\} \, . \tag{4.8}$$

The traceless elements do not close under the Jordan product, $\circ$, but one can define a symmetric product, $\star$, under which the traceless elements close as follows:[7]

$$A \star B := A \circ B - \frac{1}{(N+1)} \mathrm{tr}(A \circ B) \mathbf{1} \, ,$$

where $\circ$ is the Jordan product

$$A \circ B = \frac{1}{2}(AB + BA) \, .$$

Thus, the structure constants (d-symbols) of the traceless elements under the symmetric $\star$ product will be invariant tensors of the automorphism groups $\mathrm{Aut}(J^{\mathbb{A}}_{(1,\mathrm{N})})$ of the Jordan algebras. Denoting the traceless elements as $T_{\tilde{I}}$ ($\tilde{I} = 0, \ldots, (D-2)$) with D being the dimension of $J^{\mathbb{A}}_{(1,\mathrm{N})}$, we have

$$T_{\tilde{I}} \star T_{\tilde{J}} = d_{\tilde{I}\tilde{J}}{}^{\tilde{K}} T_{\tilde{K}} \, .$$

The d-symbols are then given by

$$d_{\tilde{I}\tilde{J}\tilde{K}} \equiv d_{\tilde{J}\tilde{K}}{}^{\tilde{L}} \tau_{\tilde{L}\tilde{I}} = \frac{1}{2} \mathrm{tr}(T_{\tilde{I}} \{T_{\tilde{J}}, T_{\tilde{K}}\}) = \mathrm{tr}(T_{\tilde{I}} \circ (T_{\tilde{J}} \circ T_{\tilde{K}})) \tag{4.9}$$

where[8]

$$\tau_{\tilde{L}\tilde{I}} = \mathrm{tr}(T_{\tilde{L}} \circ T_{\tilde{I}}) \, .$$

[6] The hermitean $(n \times n)$ matrices over the octonions do not form Jordan algebras for $n \neq 3$.

[7] Such a product was introduced among the hermitean generators of $\mathrm{SU}(N)$ by Michel and Radicati sometime ago [23]. Note that hermitean generators of $\mathrm{SU}(N)$ are in one-to-one correspondence with the traceless elements of $J^{\mathbb{C}}_N$. The "symmetric" algebras with the star product $\star$ do not have an identity element.

[8] One can choose the elements $T_{\tilde{I}}$ such that $\tau_{\tilde{I}\tilde{J}} = \delta_{\tilde{I}\tilde{J}}$ ($\tau_{\tilde{I}\tilde{J}} = -\delta_{\tilde{I}\tilde{J}}$) for two compact (noncompact) elements $T_{\tilde{I}}$ and $T_{\tilde{J}}$ and zero otherwise.

The $d_{\tilde{I}\tilde{J}\tilde{K}}$ are completely symmetric in their indices, and as $\mathrm{Aut}(J^{\mathbb{A}}_{(1,\mathrm{N})})$ acts irreducibly on the traceless elements $T_{\tilde{I}}$, the $d_{\tilde{I}\tilde{J}\tilde{K}}$ are a promising candidate for the $C_{\tilde{I}\tilde{J}\tilde{K}}$ of a unified MESGT. What remains to check, however, is whether the metrics $g_{\tilde{x}\tilde{y}}$ and $\overset{\circ}{a}_{\tilde{I}\tilde{J}}$ on the resulting scalar manifold $\mathcal{M}$ are really positive definite. As we will see, this is true if and only if the signature of η is really $(+,-,\ldots,-)$.

Let us therefore now assume that $C_{\tilde{I}\tilde{J}\tilde{K}} = d_{\tilde{I}\tilde{J}\tilde{K}}$ defines a cubic polynomial $\mathcal{V}(h) = d_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{I}} h^{\tilde{J}} h^{\tilde{K}}$ of an $\mathcal{N}=2$ MESGT. As explained in section 2, the $\tilde{n}$-dimensional scalar manifold $\mathcal{M}$ is given by the hypersurface $\mathcal{V}(h)=1$ in the auxiliary space $\mathbb{R}^{(\tilde{n}+1)}$ spanned by the $h^{\tilde{I}}$. This auxiliary space $\mathbb{R}^{(\tilde{n}+1)}$ can be identified with the traceless subspace, $J^{\mathbb{A}}_{0(1,\mathrm{N})}$, of the Jordan algebra $J^{\mathbb{A}}_{(1,\mathrm{N})}$ (i.e., the dimension, D, of $J^{\mathbb{A}}_{(1,\mathrm{N})}$ and $\tilde{n}$ are related by $D-1=\tilde{n}+1$). We will now show that a judicious choice of the traceless generators $T_{\tilde{I}}$ will bring the d-symbols $d_{\tilde{I}\tilde{J}\tilde{K}}$ (eq. (4.9)) into the canonical form (2.6). This demonstrates the positivity of the resulting metrics $g_{\tilde{x}\tilde{y}}$ and $\overset{\circ}{a}_{\tilde{I}\tilde{J}}$.

We start by noting that any point, c, on the scalar manifold $\mathcal{M}$ defines a non-zero element $h^{\tilde{I}}(c)$ in the embedding space $\mathbb{R}^{(\tilde{n}+1)}$ (it has to be non-zero, because $h^{\tilde{I}}(c)=0$ would be inconsistent with $\mathcal{V}|_{\mathcal{M}}=1$), and thus a non-trivial direction in the traceless subspace $J^{\mathbb{A}}_{0(1,\mathrm{N})}$ of the Jordan algebra $J^{\mathbb{A}}_{(1,\mathrm{N})}$. We choose our coordinates $h^{\tilde{I}}$ such that $h^{\tilde{I}}(c)=(1,0,\ldots,0)$, and, correspondingly, the generators $T_{\tilde{I}}$ such that T_0 is aligned with the non-trivial direction defined by $h^{\tilde{I}}(c)$. Note that

$$1=\mathcal{V}(h(c)) = d_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{I}}(c) h^{\tilde{J}}(c) h^{\tilde{K}}(c) = d_{000}\,. \tag{4.10}$$

A general traceless element $T_{\tilde{I}}$ in $J^{\mathbb{A}}_{(1,\mathrm{N})}$ can be represented as

$$T_{\tilde{I}} = \begin{pmatrix} x & -Y^{\dagger} \\ Y & Z \end{pmatrix}, \tag{4.11}$$

where $x\in\mathbb{R}$, Y denotes an N-dimensional column vector over $\mathbb{A}$, $Y^{\dagger}$ its hermitean conjugate and Z is a hermitean $(N\times N)$ matrix over $\mathbb{A}$ with $\mathrm{tr}(Z)=-x$. We choose

$$T_0 = \begin{pmatrix} a & 0 \\ 0 & -\frac{a}{N}\mathbf{1}_{(N)} \end{pmatrix}, \tag{4.12}$$

where a is some real number fixed to be

$$a = \left(\frac{N^2}{(N^2-1)}\right)^{\frac{1}{3}} > 0 \tag{4.13}$$

by the condition $d_{000}=1$. We use $T_M, T_N\ldots$ to denote generic elements of the form

$$T_M = \begin{pmatrix} 0 & -Y^{\dagger} \\ Y & 0 \end{pmatrix}, \tag{4.14}$$

and $T_A, T_B, \ldots$ for the generators of the type

$$T_A = \begin{pmatrix} 0 & 0 \\ 0 & Z \end{pmatrix}, \tag{4.15}$$

with $\mathrm{tr}(Z) = 0$. It is easy to see that the d-symbols of the type d_{00M} and d_{00A} vanish:

$$d_{00M} = d_{00A} = 0\,. \tag{4.16}$$

Obviously, the d-symbols of the type d_{0MN}, d_{0MA} and d_{0AB} define a metric on the subspace spanned by the h^M and h^A. This metric is negative definite, as one easily confirms by calculating the diagonal elements (no sum):

$$d_{0MM} = -a(Y^\dagger Y)\left(1 - \frac{1}{N}\right) < 0 \tag{4.17}$$

$$d_{0AA} = -\frac{a}{N}\mathrm{tr}(Z^2) < 0\,, \tag{4.18}$$

where $a > 0$ (eq. (4.13)) has been used. As $d_{0MA} = 0$, one can always go to a basis such that

$$d_{0MN} = -\frac{1}{2}\delta_{MN} \tag{4.19}$$

$$d_{0AB} = -\frac{1}{2}\delta_{AB} \tag{4.20}$$

$$d_{0MA} = 0\,. \tag{4.21}$$

Together with $d_{000} = 1$, $d_{00M} = d_{00A} = 0$, this implies that the polynomial $\mathcal{V}(h) = d_{\tilde I\tilde J\tilde K}h^{\tilde I}h^{\tilde J}h^{\tilde K}$ can always be brought to the canonical form (2.6) if one identifies $\tilde I = (0, i) = (0, M, A)$. The metrics $g_{\tilde x\tilde y}$ and $\mathring{a}_{\tilde I\tilde J}$ are thus positive definite, at least in the vicinity of the base point c. Note that this positivity requirement is precisely the point where the Minkowski signature $(1, \mathrm{N})$ of the metric η (cf. eq. (4.6)) becomes important. For metrics η with non-minkowskian signature, the diagonal elements d_{0MM} and d_{0AA} would not all be negative. Hence, only the minkowskian signatures can lead to physically acceptable unified MESGTs.

Putting everything together, we have thus shown the following: If one identifies the d-symbols (4.9) of the traceless elements of a minkowskian Jordan algebra $J^{\mathbb{A}}_{(1,\mathrm{N})}$ with the $C_{\tilde I\tilde J\tilde K}$ of a MESGT: $C_{\tilde I\tilde J\tilde K} = d_{\tilde I\tilde J\tilde K}$, one obtains a unified MESGT, in which all the vector fields transform irreducibly under the simple automorphism group $\mathrm{Aut}(J^{\mathbb{A}}_{(1,\mathrm{N})})$ of that Jordan algebra.

For $\mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ one obtains in this way three infinite families of physically acceptable unified MESGTs (one for each $N \geq 2$). For the octonionic case, the situation is a bit different. The d-symbols of the octonionic minkowskian hermitean $(N+1) \times (N+1)$-matrices with the anticommutator product all lead to positive definite metrics $g_{\tilde x\tilde y}$ and $\mathring{a}_{\tilde I\tilde J}$, i.e., to physically acceptable MESGTs. For $N \neq 2$, however, these octonionic hermitean matrix algebras are no longer Jordan algebras. Surprisingly, the automorphism groups of these octonionic algebras for $N \geq 3$ do not have the automorphism group $F_{4(-20)}$ of $J^{\mathbb{O}}_{(1,2)}$ as a subgroup [24]. Instead, the automorphism groups for $N \geq 3$ are direct product groups of the form $\mathrm{SO}(N, 1) \times G_2$. None of these two factors acts irreducibly on all the traceless elements, and hence the corresponding $\mathcal{N} = 2$ MESGTs are not unified theories. Thus, the $\mathcal{N} = 2$ MESGT defined by the exceptional minkowskian Jordan algebra $J^{\mathbb{O}}_{(1,2)}$ is the only *unified* MESGT of this infinite tower of otherwise acceptable octonionic theories.

J	D	$\mathrm{Aut}(J)$	No. of vector fields	No. of scalars
$J^{\mathbb{R}}_{(1,\mathrm{N})}$	$\frac{1}{2}(N+1)(N+2)$	$\mathrm{SO}(N,1)$	$\frac{1}{2}N(N+3)$	$\frac{1}{2}N(N+3)-1$
$J^{\mathbb{C}}_{(1,\mathrm{N})}$	$(N+1)^2$	$\mathrm{SU}(N,1)$	$N(N+2)$	$N(N+2)-1$
$J^{\mathbb{H}}_{(1,\mathrm{N})}$	$(N+1)(2N+1)$	$\mathrm{USp}(2N,2)$	$N(2N+3)$	$N(2N+3)-1$
$J^{\mathbb{O}}_{(1,2)}$	27	$F_{4(-20)}$	26	25

Table 1: List of the simple minkowskian Jordan algebras of type $J^{\mathbb{A}}_{(1,\mathrm{N})}$. The columns show, respectively, their dimensions D, their automorphism groups $\mathrm{Aut}(J^{\mathbb{A}}_{(1,\mathrm{N})})$, the number of vector fields $(\tilde{n}+1) = (D-1)$ and the number of scalars $\tilde{n} = (D-2)$ in the corresponding MESGTs.

All these results are summarized in table 1, which lists all the simple minkowskian Jordan algebras of type $J^{\mathbb{A}}_{(1,\mathrm{N})}$, their automorphism groups and the numbers of vector and scalar fields in the unified MESGT's defined by them.

As an interesting observation, one notes that the number of vector fields for the theories defined by $J^{\mathbb{R}}_{(1,3)}$, $J^{\mathbb{C}}_{(1,3)}$ and $J^{\mathbb{H}}_{(1,3)}$ are given by 9, 15 and 27, respectively. These are exactly the same numbers of vector fields one finds in the magical theories based on the norm forms of the euclidean Jordan algebras $J_3^{\mathbb{C}}$, $J_3^{\mathbb{H}}$ and $J_3^{\mathbb{O}}$, respectively. As we will show in section 6, this is not an accident: The magical MESGTs based on $J_3^{\mathbb{C}}$, $J_3^{\mathbb{H}}$ and $J_3^{\mathbb{O}}$ found in [7] are *equivalent* (i.e. the cubic polynomials $\mathcal{V}(h)$ agree) to the ones we constructed in this paper using the minkowskian algebras $J^{\mathbb{R}}_{(1,3)}$, $J^{\mathbb{C}}_{(1,3)}$ and $J^{\mathbb{H}}_{(1,3)}$, respectively. This is related to a construction of the degree 3 simple Jordan algebras $J_3^{\mathbb{C}}$, $J_3^{\mathbb{H}}$ and $J_3^{\mathbb{O}}$ in terms of the traceless elements of degree four simple Jordan algebras over $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ by Allison and Faulkner [25]. This implies, that the only known unified MESGT that is not covered by the above table, is the magical theory of [7] based on the Jordan algebra $J_3^{\mathbb{R}}$ with $(\tilde{n}+1) = 6$ vector fields and the target space $\mathcal{M} = \mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ (see section 3).

5. Unified $\mathcal{N} = 2$ Yang-Mills-Einstein supergravity theories

In the previous section, we have constructed infinitely many novel unified $\mathcal{N} = 2$ MESGTs in five dimensions by establishing a relation to a certain class of non-compact Jordan algebras. As we did not give a completeness proof, it is not clear whether these novel theories (together with the remaining magical one based on the compact Jordan algebra $J_3^{\mathbb{R}}$ found in [7]) exhaust all possible unified MESGTs in five dimensions. In order to answer that question, one would have to show that there are no further irreducible representations of simple groups with an invariant symmetric tensor of rank three that gives rise to positive metrics $g_{\tilde{x}\tilde{y}}$ and $\overset{\circ}{a}_{\tilde{I}\tilde{J}}$. We leave this as an open problem.

Instead, we will now, in this section, try to construct novel unified *Yang-Mills*-Einstein supergravity theories, i.e., theories in which all the vector fields, including the graviphoton, transform irreducibly in the adjoint representation of a simple *local* gauge group, K. If one turns off the gauge coupling of such a unified YMESGT, the local symmetry group K becomes a global symmetry group under which the vector fields still transform irreducibly. In other words, turning off the gauge coupling of a unified YMESGT yields a unified

MESGT. Conversely, any unified YMESGT can be obtained from a unified MESGT, by gauging a suitable subgroup $K \subset G$ of the global symmetry group G of the MESGT.

Let us briefly review some of the technical aspects of such a gauging [6, 5]. In the unified MESGT one starts with, the $(\tilde{n}+1)$ vector fields $A_\mu^{\tilde{I}}$ form an irreducible $(\tilde{n}+1)$-dimensional representation of the simple global symmetry group G. If one wants to construct a unified YMESGT out of this unified MESGT, one has to gauge an $(\tilde{n}+1)$-dimensional simple subgroup K of G. For this to be possible, the $(\tilde{n}+1)$-dimensional representation of G has to reduce to the adjoint representation of K under the restriction $G \rightarrow K$:

$$(\tilde{\mathbf{n}}+\mathbf{1})_{\mathbf{G}} \rightarrow \mathrm{adjoint}(K)\,. \tag{5.1}$$

The only fields in the $\mathcal{N}=2$ MESGT that transform nontrivially under K are the scalar fields $\varphi^{\tilde{x}}$, the spinor fields $\lambda^{i\tilde{a}}$ and the vector fields $A_\mu^{\tilde{I}}$, $(\tilde{I}=1,\ldots,\mathrm{dim}K)$. The K-covariantization is then achieved by first replacing the corresponding derivatives/field strengths by their K-gauge covariant counterparts:

$$\begin{aligned}
\partial_\mu\varphi^{\tilde{x}} &\longrightarrow \mathcal{D}_\mu\varphi^{\tilde{x}} \equiv \partial_\mu\varphi^{\tilde{x}} + gA_\mu^{\tilde{I}}K_{\tilde{I}}^{\tilde{x}} \\
\nabla_\mu\lambda^{i\tilde{a}} &\longrightarrow \mathcal{D}_\mu\lambda^{i\tilde{a}} \equiv \nabla_\mu\lambda^{i\tilde{a}} + gA_\mu^{\tilde{I}}L_{\tilde{I}}^{\tilde{a}\tilde{b}}\lambda^{i\tilde{b}} \\
F_{\mu\nu}^{\tilde{I}} &\longrightarrow \mathcal{F}_{\mu\nu}^{\tilde{I}} \equiv F_{\mu\nu}^{\tilde{I}} + gf_{\tilde{J}\tilde{K}}^{\tilde{I}}A_\mu^{\tilde{J}}A_\nu^{\tilde{K}}\,.
\end{aligned} \tag{5.2}$$

Here, g denotes the coupling constant of K, $K_{\tilde{I}}^{\tilde{x}}$ are the Killing vectors that generate the subgroup $K \subset G$ of isometries of the scalar manifold $\mathcal{M}$ (cf. [6]), $L_{\tilde{I}}^{\tilde{a}\tilde{b}}$ are the (scalar field dependent) K-transformation matrices of the fermions $\lambda^{i\tilde{a}}$ (cf. [6, 5]), and $f_{\tilde{I}\tilde{J}}^{\tilde{K}}$ are the structure constants of K. The proper gauge-covariantization of the $F\wedge F\wedge A$-term in (2.2) leads to a Chern Simons term, i.e.,

$$\frac{e^{-1}}{6\sqrt{6}}C_{\tilde{I}\tilde{J}\tilde{K}}\varepsilon^{\mu\nu\rho\sigma\lambda}F_{\mu\nu}^{\tilde{I}}F_{\rho\sigma}^{\tilde{J}}A_\lambda^{\tilde{K}}$$

has to be replaced by

$$\begin{aligned}
\frac{e^{-1}}{6\sqrt{6}}C_{\tilde{I}\tilde{J}\tilde{K}}\varepsilon^{\mu\nu\rho\sigma\lambda}\Big\{&F_{\mu\nu}^{\tilde{I}}F_{\rho\sigma}^{\tilde{J}}A_\lambda^{\tilde{K}} + \frac{3}{2}gF_{\mu\nu}^{\tilde{I}}A_\rho^{\tilde{J}}\left(f_{\tilde{L}\tilde{M}}^{\tilde{K}}A_\sigma^{\tilde{L}}A_\lambda^{\tilde{M}}\right) + \\
&+\frac{3}{5}g^2\left(f_{\tilde{N}\tilde{P}}^{\tilde{J}}A_\nu^{\tilde{N}}A_\rho^{\tilde{P}}\right)\left(f_{\tilde{L}\tilde{M}}^{\tilde{K}}A_\sigma^{\tilde{L}}A_\lambda^{\tilde{M}}\right)A_\mu^{\tilde{I}}\Big\}\,.
\end{aligned} \tag{5.3}$$

Supersymmetry is broken by these replacements. In order to restore it, one has to add a Yukawa-like term to the (covariantized) lagrangian [6, 5]:

$$\mathcal{L}' = -\frac{i}{2}g\bar{\lambda}^{i\tilde{a}}\lambda_i^{\tilde{b}}K_{\tilde{I}[\tilde{a}}h_{\tilde{b}]}^{\tilde{I}}\,, \tag{5.4}$$

where $h_{\tilde{b}}^{\tilde{I}}$ is essentially the derivative of $h^{\tilde{I}}$ with respect to the scalar fields $\varphi^{\tilde{x}}$ (see [7] for details). The covariantized supersymmetry transformation laws remain unmodified. Note that (in the absence of tensor multiplets) such an $\mathcal{N}=2$ Yang-Mills-Einstein supergravity theory has no scalar potential, i.e., the ground states are Minkowski space-times.

5.1 The complete list of 5D, $\mathcal{N}=2$ unified Yang-Mills-Einstein supergravity theories

While we did not give a completeness proof of our list of unified MESGTs, we are able to give a complete list of all possible unified $\mathcal{N}=2$ YMESGTs in five dimensions. This completeness proof is actually rather simple. In a unified YMESGT, the vector fields $A_\mu^{\tilde{I}}$ (and with them the embedding coordinates $h^{\tilde{I}}$) transform, by definition, in the adjoint representation of a simple group K. The tensor $C_{\tilde{I}\tilde{J}\tilde{K}}$ of the supergravity theory then has to be a symmetric cubic invariant of the adjoint representation of K. Of all the simple groups, only the unitary groups $\mathrm{SU}(N)$ $(N\geq 3)$ and their different real forms have such an invariant, namely the Gell-Mann d-symbols $d_{\tilde{I}\tilde{J}\tilde{K}}=1/2\mathrm{Tr}(T_{\tilde{I}}\{T_{\tilde{J}},T_{\tilde{K}}\})$, where $T_{\tilde{I}}$ denote the generators of $\mathrm{SU}(N)$ or one of its different real forms. Just as we did in section 4, one can then show that only the groups of the type $\mathrm{SU}(N,1)$ $(N\geq 2)$ can lead to positive metrics $g_{\tilde{x}\tilde{y}}$ and $\overset{\circ}{a}_{\tilde{I}\tilde{J}}$, because only their d-symbols can be transformed to the canonical basis (2.6). Comparing with the d-symbols (4.9) for the Jordan algebra $J^{\mathbb{C}}_{(1,\mathrm{N})}$, and taking into account the isomorphism between its traceless elements and the generators of $\mathrm{SU}(1,\mathrm{N})$, we see that the unified YMESGTs we have just found arise from the gauging of the unified MESGTs related to $J^{\mathbb{C}}_{(1,\mathrm{N})}$. As the magical MESGT corresponding to the euclidean algebra $J_3^{\mathbb{H}}$ is equivalent to the one obtained from $J^{\mathbb{C}}_{(1,3)}$ (cf the discussion at the end of section 4 and section 6), this shows that *all* unified $\mathcal{N}=2$ MESGTs in five dimensions are obtained by gauging the full $\mathrm{SU}(N,1)$ automorphism groups of the unified MESGTs defined by the Jordan algebras $J^{\mathbb{C}}_{(1,\mathrm{N})}$ (see table 1). It is easy to convince oneself that the other novel families of unified MESGTs cannot lead to unified YMESGTs.

Let us close this subsection with a few remarks on the physical properties of the unified YMESGTs. In gauging the full $\mathrm{SU}(N,1)$ symmetry, the $F\wedge F\wedge A$ term gets replaced by the Chern-Simons form (5.3) of $\mathrm{SU}(N,1)$. Since the fifth homotopy group Π_5 of $\mathrm{SU}(N,1)$ is the set of integers $\mathbb{Z}$:

$$\Pi_5(\mathrm{SU}(N,1))=\Pi_5(\mathrm{U}(N))=\Pi_5(\mathrm{SU}(N))=\mathbb{Z}\,,$$

the quantum gauge invariance under large gauge transformations require that the dimensionless ratio g^3/κ of the third power of the non-abelian gauge coupling constant g and the gravitational constant κ must be quantized [5].

As mentioned earlier, a YMESGT without tensor or hypermultiplets does not have a scalar potential. This means that the vacuum expectation values (vevs) of the scalar fields $\varphi^{\tilde{x}}$ (or equivalently the vevs of the fields $h^{\tilde{I}}(\varphi)$) are not fixed. A vev $\langle h^{\tilde{I}}\rangle$ corresponding to a compact direction in the Lie algebra of $\mathrm{SU}(N,1)$, can always be chosen as the base point c of the canonical basis. The little group of the base point corresponding to the element T_0 of $J^{\mathbb{C}}_{0(1,\mathrm{N})}$ is $\mathrm{U}(N)\subset\mathrm{SU}(1,\mathrm{N})$. This is the remaining unbroken gauge group in the vacuum. Under this unbroken $\mathrm{U}(N)$, the $N(N+2)-1$ scalar fields decompose as $N^{(+1)}\oplus\bar{N}^{(-1)}\oplus(N^2-1)^{(0)}$, while the $N(N+2)$ vector fields decompose as $1^{(0)}\oplus N^{(+1)}\oplus\bar{N}^{(-1)}\oplus(N^2-1)^{(0)}$. The singlet is to be identified with the graviphoton, which is thus no longer 'unified' with the other vector fields under the action of $\mathrm{U}(N)$. This was to be expected, as the non-compact gauge symmetries, which connect the graviphoton with the other vector fields, have to be broken in any vacuum, as required by unitarity[9]

The gauge fields associated with the non-compact generators eat the scalar fields in the $N^{(+1)} \oplus \bar{N}^{(-1)}$ and become massive vector fields transforming in the $N^{(+1)} \oplus \bar{N}^{(-1)}$ of $\mathrm{U}(N)$. Due to the extra Yukawa coupling term $\mathcal{L}'$ (eq. (5.4)) introduced in the lagrangian to restore supersymmetry after the gauging, the spin 1/2 fields in the $N^{(+1)} \oplus \bar{N}^{(-1)}$ of $\mathrm{U}(N)$ also become massive. Together with the massive vector fields, they form massive BPS vector multiplets. The central charge of these BPS multiplets is generated by the $\mathrm{U}(1)$ factor in $\mathrm{U}(N)$, which is gauged by the graviphoton. The massless spectrum thus consists of $\mathcal{N} = 2$ $\mathrm{SU}(N)$ super Yang-Mills coupled to $\mathcal{N} = 2$ supergravity.

5.2 Coupling of tensor fields to unified Yang-Mills-Einstein supergravity theories

As we have seen in the previous section, the only possible gauge groups of 5D unified YMESGTs are of the form $\mathrm{SU}(N,1)$ with $N \geq 2$. All these theories are obtained by gauging the full $\mathrm{SU}(N,1)$ automorphism groups of the unified MESGTs defined by the minkowskian Jordan algebras $J^{\mathbb{C}}_{(1,\mathrm{N})}$. A natural question to ask now is whether there are gaugings of the other novel unified MESGTs based on the Jordan algebras $J^{\mathbb{R}}_{(1,\mathrm{N})}$, $J^{\mathbb{H}}_{(1,\mathrm{N})}$ and $J^{\mathbb{O}}_{(1,2)}$ that come as close as possible to what we called 'unified YMESGTs'. As we have already stated, one cannot gauge these theories such that all the vector fields of the ungauged theory become the gauge fields of a simple gauge group. As was pointed out in [9], however, if the $(\tilde{n}+1)$-dimensional representation of the global symmetry group G of a MESGT decomposes under a subgroup $K \subset G$ as

$$(\tilde{\mathbf{n}} + \mathbf{1})_{\mathbf{G}} \rightarrow \mathrm{adjoint}(K) \oplus \text{non-singlets}(K)\,, \tag{5.5}$$

one can sometimes gauge K by turning the non-singlet vector fields into self-dual tensor fields of the type first described in [26]. To this end, one splits the index $\tilde{I}$ of the vector fields into two sets: $\tilde{I} \rightarrow (I, M)$, where $I, J, \ldots$ correspond to the adjoint of K, and $M, N, \ldots$ label the non-singlets outside the adjoint. The vector fields A^I_μ then play the role of the gauge fields of K, whereas the non-singlet vector fields A^M_μ have to be converted to 2-form fields $B^M_{\mu\nu}$. The technical details of this kind of gauging can be found in [9]. One finds that the gauging of K is possible only if the non-singlets transform in a symplectic representation of the gauge group K, and the $C_{\tilde{I}\tilde{J}\tilde{K}}$ components of the type C_{MIJ} and C_{MNP} vanish [9].[10] Furthermore, one finds that the gauging in the presence of tensor fields introduces a scalar potential (as well as Yukawa couplings). The scalar potential is manifestly positive semidefinite [9].

[9] In any given Minkowski vacuum with constant vevs $\langle h^{\tilde{I}} \rangle$, the physical graviphoton, i.e., the linear combination of vector fields that appears in the gravitino supersymmetry variation, is given by $A_\mu = \langle h_{\tilde{I}} \rangle A^{\tilde{I}}_\mu$, where $h_{\tilde{I}} \equiv \overset{\circ}{a}_{\tilde{I}\tilde{J}} h^{\tilde{J}}$ [7]. It is easy to see that this linear combination is automatically invariant under the transformations (3.1). What we mean when we say that the 'unifying' symmetry maps the 'graviphoton' to the other vector fields is that it acts irreducibly on all the vector fields $A^{\tilde{I}}_\mu$.

[10] As was pointed out in [27], there are cases where the coefficients C_{MIJ} might be non-zero. For *simple* (i.e., unifying) gauge groups K, however, this cannot be the case.

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One can thus try to gauge some of the other unified MESGTs such that the gauge group K is of the form $\mathrm{SU}(N,1)$, and all the vector fields outside the adjoint of K are converted to tensor fields. Such a theory can then be interpreted as a unified YMESGT coupled to tensor multiplets. In a vacuum of such a theory, the gauge group is again broken down to its maximal compact subgroup $\mathrm{U}(N)$, and the massless spectrum consists of an $\mathrm{SU}(N)$ super Yang-Mills multiplet plus the 5D, $\mathcal{N}=2$ supergravity multiplet. The massive part of the spectrum will consist of $2N$ massive BPS vector multiplets in the $(N \oplus \bar{N})$ of $\mathrm{SU}(N)$ plus the tensor multiplets, which form massive BPS tensor multiplets [28, 29, 17]. The number of these tensor multiplets depends on the particular theory under consideration.[11] Let us first start with the family of theories based on the Jordan algebras $J^{\mathbb{H}}_{(1,\mathrm{N})}$. Under the automorphism group $\mathrm{USp}(2N,2)$ of $J^{\mathbb{H}}_{(1,\mathrm{N})}$, the traceless elements corresponding to the vector fields in the MESGT transform in the anti-symmetric symplectic traceless representation

$$J^{\mathbb{H}}_{0(1,\mathrm{N})} \Longleftrightarrow (2N^2+3N)\,.$$

Since the Jordan algebra $J^{\mathbb{H}}_{(1,\mathrm{N})}$ contains the complex Jordan algebra $J^{\mathbb{C}}_{(1,\mathrm{N})}$ as a subalgebra, one can gauge an $\mathrm{SU}(N,1)$ subgroup of $\mathrm{USp}(2N,2)$ with the remaining $N(N+1)$ vector fields dualized to tensor fields transforming in the reducible symplectic representation

$$\frac{N(N+1)}{2} \oplus \overline{\frac{N(N+1)}{2}}$$

of $\mathrm{SU}(N,1)$ for $N \geq 2$.

In the family of unified MESGTs based on $J^{\mathbb{R}}_{(1,\mathrm{N})}$, the vector fields transform in the symmetric tensor representation of the automorphism group $\mathrm{SO}(N,1)$ of $J^{\mathbb{R}}_{(1,\mathrm{N})}$. Now $J^{\mathbb{R}}_{(1,\mathrm{N})}$ is a subalgebra of $J^{\mathbb{C}}_{(1,\mathrm{N})}$ and its traceless elements correspond to the generators belonging to the coset space $\mathrm{SU}(N,1)/\,\mathrm{SO}(N,1)$. In this case, the maximal unifying gauge groups of the type $\mathrm{SU}(1,M)$ are smaller than the maximal possible non-abelian gauge groups, which turn out to be compact. More concretely, for $N=2n$ with $N>3$ one can gauge the $\mathrm{U}(n)$ subgroup of $\mathrm{SO}(2n,1)$ with the remaining vector fields dualized to tensor fields transforming in the reducible symplectic representation

$$\frac{n(n+1)}{2} \oplus \overline{\frac{n(n+1)}{2}}$$

[11] Interestingly, if, in any of the above-mentioned theories, one chooses to gauge only the $\mathrm{U}(N)$ subgroup of $\mathrm{SU}(1,\mathrm{N})$, the $2N$ massive BPS vector multiplets are replaced by $2N$ massive BPS tensor multiplets. For the theory based on $J^{\mathbb{C}}_{(1,\mathrm{N})}$ with $N=5$, this kind of gauging would, for example, lead to a minimal 5D, $\mathcal{N}=2$ supersymmetric $\mathrm{SU}(5)$ GUT model with 5D tensor multiplets in the $(5 \oplus \bar{5})$ of $\mathrm{SU}(5)$. A very similar model was considered in ref. [18], where the $(5 \oplus \bar{5})$ tensor multiplets were interpreted as the $\mathrm{SU}(5)$ multiplets that contain the Standard Model Higgs fields. The essential difference between the model of [18] and our construction is an additional $\mathrm{SU}(5)$ singlet vector multiplet, which was introduced in [18], but does not occur in our case. It should also be noted that 5D $\mathrm{SU}(5)$ models with tensor multiplets in the $(5 \oplus \bar{5})$ automatically have an additional $\mathrm{U}(1)$ factor in the gauge group under which the tensor multiplets are charged (see [17] for a general proof of this statement). The corresponding gauge field can be identified with the graviphoton. Note finally that, for any of the theories of the type described above, one can always choose to gauge only a subgroup of the type $\mathrm{SU}(1,M)$ with $M<N$. This would simply result in more tensor fields.

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of U(n). For $N = 2n+1$ ($N > 3$) one can gauge the U(n) subgroup of SO($2n+1, 1$) with the tensor fields in the reducible representation

$$(n \oplus \bar{n}) \bigoplus \left(\frac{n(n+1)}{2} \oplus \overline{\frac{n(n+1)}{2}} \right) \bigoplus (1 \oplus \bar{1})\,.$$

As the gauge groups are not of the type SU($1, M$), the Yang-Mills sectors of these maximally gauged theories are, of course, no longer 'unified'.

Finally, the novel exceptional octonionic MESGT based on $J^{\mathbb{O}}_{(2,1)}$ has 26 vector fields transforming irreducibly under its automorphism group $F_{4(-20)}$. In this case, one can gauge the SU($2, 1$) subgroup with the remaining vector fields replaced by tensor fields transforming in the reducible symplectic representation

$$(3 \oplus \bar{3}) \bigoplus (3 \oplus \bar{3}) \bigoplus (3 \oplus \bar{3})\,.$$

6. The geometry of the novel unified MESGTs and the remarkable isoparametric hypersurfaces of Elie Cartan

In this section we take a first look at the geometry of the novel scalar manifolds we found in this paper. To begin with, let us first isolate the scalar manifolds that are symmetric or homogeneous spaces. From the known classification of $\mathcal{N} = 2$ MESGTs whose scalar manifolds are symmetric or homogeneous spaces we expect the scalar manifolds of the novel unified MESGTs given in the section 4 to be neither symmetric nor homogeneous, in general. More precisely, the symmetries of the $\mathcal{N} = 2$ MESGTs whose scalar manifolds are homogeneous, but not symmetric, as classified in [10, 13], are such that they cannot coincide with any of the novel unified theories listed in section 4, since their vector fields do not transform irreducibly under a simple noncompact symmetry group. Thus, if there are homogeneous space examples among the novel theories, they also have to be symmetric spaces. We already showed in section 3, that, among the theories with symmetric target spaces, only the four magical MESGTs are unified MESGTs. Thus, this only leaves open the possibility that some of the novel unified MESGTs may be equivalent to some of the magical theories. The magical MESGTs are all defined by cubic forms that are norm forms of simple Jordan algebras of degree three, so to answer this question we need to check if any of the cubic forms of the novel theories coincide with the norm forms of some simple Jordan algebra of degree three. Remarkably, and as was already announced at the end of section 4, we find that the cubic forms defined by the structure constants of the traceless elements of the minkowskian Jordan algebras $J^{\mathbb{R}}_{(1,3)}, J^{\mathbb{C}}_{(1,3)}$ and $J^{\mathbb{H}}_{(1,3)}$ coincide with the norm forms of the simple Jordan algebras $J^{\mathbb{C}}_{(3)}, J^{\mathbb{H}}_{(3)}$ and $J^{\mathbb{O}}_{(3)}$, respectively. To see this, consider the example of $J^{\mathbb{R}}_{(1,3)}$, a general traceless element of which can be parameterized as

$$M = \begin{pmatrix} 3a & x & y & z \\ -x & -a+b & u & v \\ -y & u & -a+c & w \\ -z & v & w & -a-b-c \end{pmatrix}.$$

On the other hand, consider a general element of $J^{\mathbb{C}}_{(3)}$

$$J = \begin{pmatrix} 2a+b & u+iz & v-iy \\ u-iz & 2a+c & w+ix \\ v+iy & w-ix & 2a-b-c \end{pmatrix} .$$

One finds that

$$\det J = \frac{1}{3}\mathrm{tr} M^3 ,$$

proving the equivalence of the theories defined by the corresponding cubic forms. One can similarly show the equivalence of the cubic forms over the traceless elements of $J^{\mathbb{C}}_{(1,3)}$ and $J^{\mathbb{H}}_{(1,3)}$ with the norm forms of $J^{\mathbb{H}}_{(3)}$ and $J^{\mathbb{O}}_{(3)}$, respectively [25]. Thus the MESGTs defined by these cubic forms have the reduced structure groups of the simple Jordan algebras $J^{\mathbb{C}}_{(3)}$, $J^{\mathbb{H}}_{(3)}$ and $J^{\mathbb{O}}_{(3)}$ as their enlarged hidden symmetry groups. More specifically, the MESGT defined by the structure constants of the traceless elements of $J^{\mathbb{R}}_{(1,3)}$ has the symmetry group $\mathrm{SL}(3,\mathbb{C})$ which has the automorphism group $\mathrm{SO}(3,1)$ of $J^{\mathbb{R}}_{(1,3)}$ as a subgroup. The MESGT theory defined by the structure constants of $J^{\mathbb{C}}_{(1,3)}$ has the symmetry group $\mathrm{SU}^*(6)$ which has the automorphism group $\mathrm{SU}(3,1)$ of $J^{\mathbb{C}}_{(1,3)}$ as a subgroup. Finally, the theory defined by the structure constants of the traceless elements of $J^{\mathbb{H}}_{(1,3)}$ has the symmetry group $E_{6(-26)}$, which has the automorphism group $\mathrm{USp}(6,2)$ of $J^{\mathbb{H}}_{(1,3)}$ as a subgroup.[12] To sum up, only three of our novel theories have homogeneous scalar manifolds, and they are equivalent to three of the theories found in [7].

Thus, almost all of the novel theories have non-homogeneous scalar manifolds. Nevertheless, some of these non-homogeneous spaces can be related to homogeneous spaces in an interesting way that resembles an old construction by E. Cartan [8]. More explicitly, consider the scalar manifolds of the novel MESGTs defined by the structure constants of the minkowskian Jordan algebras $J^{\mathbb{A}}_{(1,2)}$ of degree three. They turn out to be submanifolds of "minkowskian" versions of the target spaces of the magical supergravity theories defined by the norm forms of euclidean Jordan algebras $J^{\mathbb{A}}_3$. These submanifolds are themselves foliated by hypersurfaces that are the non-compact analogs of the remarkable families of isoparametric hypersurfaces in 4, 7, 13 and 25 dimensions that were studied by Elie Cartan long time ago [8].

To show this connection with the work of Cartan, consider the general element of the Jordan algebra $J^{\mathbb{A}}_{(1,2)}$[13]

$$J = \begin{pmatrix} \sqrt{3}x_4 - x_0 - 2\cos t & \sqrt{3}x_3 & -\sqrt{3}\bar{x}_2 \\ \sqrt{3}\bar{x}_3 & -\sqrt{3}x_4 - x_0 - 2\cos t & -\sqrt{3}x_1 \\ \sqrt{3}x_2 & \sqrt{3}\bar{x}_1 & 2x_0 - 2\cos t \end{pmatrix} , \tag{6.1}$$

where x_1, x_2, x_3 are elements of the division algebra $\mathbb{A}$ and x_0, x_4 and t are some real numbers. The cubic norm of J is given by

$$\begin{aligned} N(J) \;=\; & -8\cos^3 t + 6\cos t\left[-x_1\bar{x}_1 - x_2\bar{x}_2 + x_3\bar{x}_3 + x_0^2 + x_4^2\right] + 2x_0^3 - \\ & -3x_0\left[2x_4^2 + 2x_3\bar{x}_3 + x_1\bar{x}_1 + x_2\bar{x}_2\right] + 3\sqrt{3}x_4[x_1\bar{x}_1 - x_2\bar{x}_2] - 6\sqrt{3}\mathrm{Re}(x_3x_2x_1) , \end{aligned} \tag{6.2}$$

[12] $\mathrm{USp}(6,2)$ has $\mathrm{USp}(6) \times \mathrm{USp}(2)$ as a maximal compact subgroup.

[13] Our labelling follows that of Cartan, even though he did not use the language of Jordan algebras.

where $\mathrm{Re}(x)$ stands for the real part of an element x of $\mathbb{A}$. Now the hypersurfaces defined by the condition

$$N(J) = 1$$

are the coset spaces:

$$\begin{aligned}
\frac{\mathrm{SL}(3,\mathbb{R})}{\mathrm{SL}(2,\mathbb{R})} &\Leftrightarrow J^{\mathbb{R}}_{(1,2)} \\
\frac{\mathrm{SL}(3,\mathbb{C})}{\mathrm{SU}(2,1)} &\Leftrightarrow J^{\mathbb{C}}_{(1,2)} \\
\frac{\mathrm{SU}^*(6)}{\mathrm{USp}(4,2)} &\Leftrightarrow J^{\mathbb{H}}_{(1,2)} \\
\frac{E_{6(-26)}}{F_{4(-20)}} &\Leftrightarrow J^{\mathbb{O}}_{(1,2)} \,.
\end{aligned}$$

Note that in the corresponding manifolds of the magical MESGTs defined by euclidean Jordan algebras of degree three (eq. (4.5)), the reduced structure group is the same, but the automorphism group is its maximal *compact* subgroup. If one were to use the cubic norm form of the above minkowskian Jordan algebra of degree three to construct a MESGT, the kinetic energy terms of the vector fields as well as those of the scalar fields would not be positive definite, rendering these theories unphysical. For the theories constructed in section 4, on the other hand, the cubic polynomials associated with the novel MESGTs are given by

$$\mathrm{tr} J_0^3 \,,$$

where J_0 is a generic traceless element of $J^{\mathbb{A}}_{(1,2)}$. Using the parametrization

$$J_0 = \begin{pmatrix} \sqrt{3}x_4 - x_0 & \sqrt{3}x_3 & -\sqrt{3}\bar{x}_2 \\ \sqrt{3}\bar{x}_3 & -\sqrt{3}x_4 - x_0 & -\sqrt{3}x_1 \\ \sqrt{3}x_2 & \sqrt{3}\bar{x}_1 & 2x_0 \end{pmatrix},$$

one obtains

$$\begin{aligned}
\frac{1}{3}\mathrm{tr}(J_0^3) &= 2x_0^3 - 3x_0(2x_4^2+2x_3\bar{x}_3+x_1\bar{x}_1+x_2\bar{x}_2)+3\sqrt{3}x_4(x_1\bar{x}_1 - x_2\bar{x}_2) - 6\sqrt{3}\mathrm{Re}(x_3x_2x_1) \\
&= \det(J) + 8\cos^3 t - 6\cos t\left[-x_1\bar{x}_1 - x_2\bar{x}_2 + x_3\bar{x}_3 + x_0^2 + x_4^2\right] . \qquad (6.3)
\end{aligned}$$

If we now impose the constraint:

$$-x_1\bar{x}_1 - x_2\bar{x}_2 + x_3\bar{x}_3 + x_0^2 + x_4^2 = \frac{4}{3}\cos^2 t \qquad (6.4)$$

we find that

$$\frac{1}{3}\mathrm{tr}(J_0^3) = \det(J)|_{\mathrm{constraint}(6.4)} \,.$$

Noting that for a fixed value of t the equation (6.4) defines a non-compact "hypersphere" and comparing our formulas with those of Elie Cartan, we see that the equation

$$\det J = \text{constant}$$

subject to the constraint $-x_1\bar{x}_1 - x_2\bar{x}_2 + x_3\bar{x}_3 + x_0^2 + x_4^2 = \frac{4}{3}\cos^2 t$ go over to his equations if we replace $x_1\bar{x}_1$ and $x_2\bar{x}_2$ with their negatives. Thus, the scalar manifolds of the novel MESGTs defined by $J^{\mathbb{A}}_{(1,2)}$ are the noncompact analogs of the manifolds studied by Cartan. Cartan showed that the remarkable compact hypersurfaces he studied in 4, 7, 13 and 25 dimensions exhaust the list of isoparametric hypersurfaces in spheres with three distinct curvatures. Since the scalar manifolds of the novel unified theories are given by the condition

$$\mathrm{tr}(J_0^3) = \text{constant}\,,$$

we see that they are foliated by the noncompact analogues of these remarkable hypersurfaces.

The hypersurfaces studied by Cartan are related to the homogeneous spaces [30]

$$\begin{gathered}\frac{\mathrm{SO}(3)}{\mathbb{Z}_2^2}\\ \frac{\mathrm{SU}(3)}{T^2}\\ \frac{\mathrm{USp}(6)}{\mathrm{SU}(2)^3}\\ \frac{F_4}{\mathrm{Spin}(8)}\,.\end{gathered} \tag{6.5}$$

The corresponding hypersurfaces in the scalar manifolds of MESGTs defined by $J^{\mathbb{A}}_{(1,2)}$ are related to certain noncompact versions of the above homogeneous manifolds. The detailed study of these manifolds and their extension to manifolds of theories defined by higher dimensional Jordan algebras will be left for future work.

7. Conclusions

Extended supergravity theories often exhibit extra non-compact bosonic symmetries that are not part of the underlying R-symmetry groups. These extra symmetries can connect fields with the same Lorentz quantum numbers even if these fields originate from different types of supermultiplets. In particular, there can exist non-compact bosonic symmetries that can mediate between vector fields from vector multiplet sectors and vector fields from the supergravity sector. In some cases, all the vector fields transform irreducibly under a single simple symmetry group of this type. Alluding to the conventional GUT terminology, we called such supergravity theories 'unified' MESGTs or 'unified' YMESGTs depending on whether the simple, 'unifying' symmetry group is a global or a local symmetry of the theory. For 5D, $\mathcal{N}=4$ supergravity, such a unifying symmetry is impossible, as there would always be at least one singlet vector field in the $\mathcal{N}=4$ supergravity multiplet, no matter how the simple symmetry group is chosen [31]. Remarkably, for 5D, $\mathcal{N}=2$ supergravity, such unified MESGTs and YMESGTs do exist.

The general 5D, $\mathcal{N}=2$ MESGTs are in one-to-one correspondence with cubic polynomials $\mathcal{V}(h) = C_{\tilde{I}\tilde{J}\tilde{K}}h^{\tilde{I}}h^{\tilde{J}}h^{\tilde{K}}$ that can be brought to the canonical form (2.6), which ensures positive kinetic terms in the action. A unified MESGT is obtained, when such an admissible set of coefficients $C_{\tilde{I}\tilde{J}\tilde{K}}$ forms an invariant symmetric tensor of an irreducible representation of a simple group.

In this paper, we have found infinitely many examples of such tensors by using the language of Jordan algebras. There are essentially two ways a Jordan algebra can give rise to a cubic polynomial of the type $\mathcal{V}(h) = C_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{I}} h^{\tilde{J}} h^{\tilde{K}}$:

(i) Every Jordan algebra J has a norm form $N : J \to \mathbb{R}$. When this norm form is cubic (i.e., when the Jordan algebra is of degree $p = 3$), it defines a cubic polynomial $\mathcal{V}(h) = N$. In general, such a cubic polynomial can not be brought to the canonical form (2.6). The Jordan algebras for which this *is* possible, are precisely the *euclidean* Jordan algebras of degree three (cf. items (i) and (ii) in section 3 or section 4). All these euclidean Jordan algebras of degree three define admissible MESGTs, however, only the four *simple* ones (i.e. the hermitean (3×3) matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$) lead to *unified* MESGTs. These unified MESGTs were constructed in [7].

(ii) Being completely symmetric, the structure constants of a Jordan algebra also define a cubic polynomial. By construction, this polynomial is invariant under the automorphism group of the Jordan algebra, but as this automorphism group does not act irreducibly on the Jordan algebra generators, such a polynomial cannot give rise to a *unified* MESGT. In this paper, we showed, that one can nevertheless obtain unified MESGTs from these structure constants, provided that one restricts oneself to the traceless elements and takes J to be equal to any of the *minkowskian* Jordan algebras $J_{(1,\mathrm{N})}^{\mathbb{A}=\mathbb{R},\mathbb{C},\mathbb{H}}$ or $J_{(1,2)}^{\mathbb{O}}$. This way, we obtained three infinite families and one novel exceptional unified MESGT. Interestingly, the novel theories based on $J_{(1,3)}^{\mathbb{R}}$, $J_{(1,3)}^{\mathbb{C}}$ $J_{(1,3)}^{\mathbb{H}}$ are equivalent to the ones based on $J_3^{\mathbb{C}}$, $J_3^{\mathbb{H}}$, $J_3^{\mathbb{O}}$, respectively.

The unified MESGTs based on $J_{(1,\mathrm{N})}^{\mathbb{C}}$ can all be turned into unified YMESGTs by gauging the full automorphism groups $\mathrm{SU}(1,\mathrm{N})$. These theories exhaust all possible unified YMESGTs in 5D and include the one discovered in [6].

As a by-product of our considerations, we found that the scalar manifolds based on $J_{(1,2)}^{\mathbb{A}}$ are foliated by certain noncompact analogues of the isoparametric hypersurfaces in spaces of constant curvature studied by E. Cartan [8] long time ago.

The existence of unified $\mathcal{N} = 2$ MESGTs and YMESGTs is not special only to five dimensions. They also exist in four dimensions. The classification of the four dimensional unified MESGTs and YMESGTs, as well as the higher-dimensional origin of our theories will be left for future work.

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References

[1] B. de Wit and H. Nicolai, $N = 8$ *supergravity*, *Nucl. Phys.* **B 208** (1982) 323.

[2] U. Amaldi, W. de Boer and H. Furstenau, *Comparison of grand unified theories with electroweak and strong coupling constants measured at lep*, *Phys. Lett.* **B 260** (1991) 447;
C. Giunti, C.W. Kim and U.W. Lee, *Running coupling constants and grand unification models*, *Mod. Phys. Lett.* **A 6** (1991) 1745;
J.R. Ellis, S. Kelley and D.V. Nanopoulos, *Probing the desert using gauge coupling unification*, *Phys. Lett.* **B 260** (1991) 131;
P. Langacker and M.-X. Luo, *Implications of precision electroweak experiments for m_T, ρ_0, $\sin^2\theta_W$ and grand unification*, *Phys. Rev.* **D 44** (1991) 817.

[3] Y. Kawamura, *Gauge symmetry reduction from the extra space S^1/Z_2*, *Prog. Theor. Phys.* **103** (2000) 613 [hep-ph/9902423]; *Triplet-doublet splitting, proton stability and extra dimension*, *Prog. Theor. Phys.* **105** (2001) 999 [hep-ph/0012125]; *Split multiplets, coupling unification and extra dimension*, *Prog. Theor. Phys.* **105** (2001) 691 [hep-ph/0012352];
G. Altarelli and F. Feruglio, SU(5) *grand unification in extra dimensions and proton decay*, *Phys. Lett.* **B 511** (2001) 257 [hep-ph/0102301];
A.B. Kobakhidze, *Proton stability in TeV-scale guts*, *Phys. Lett.* **B 514** (2001) 131 [hep-ph/0102323];
L.J. Hall and Y. Nomura, *Gauge unification in higher dimensions*, *Phys. Rev.* **D 64** (2001) 055003 [hep-ph/0103125];
A. Hebecker and J. March-Russell, *A minimal $S^1/(Z_2 \times Z_2')$ orbifold GUT*, *Nucl. Phys.* **B 613** (2001) 3 [hep-ph/0106166];
R. Dermisek and A. Mafi, SO(10) *grand unification in five dimensions: proton decay and the mu problem*, *Phys. Rev.* **D 65** (2002) 055002 [hep-ph/0108139].

[4] M. de Roo, *Matter coupling in $N = 4$ supergravity*, *Nucl. Phys.* **B 255** (1985) 515.

[5] M. Günaydin, G. Sierra and P.K. Townsend, *Quantization of the gauge coupling constant in a five- dimensional Yang-Mills/Einstein supergravity theory*, *Phys. Rev. Lett.* **53** (1984) 322.

[6] M. Günaydin, G. Sierra and P.K. Townsend, *Gauging the $D = 5$ Maxwell-Einstein supergravity theories: more on Jordan algebras*, *Nucl. Phys.* **B 253** (1985) 573.

[7] M.Günaydin, G.Sierra and P.K.Townsend, *The geometry of $N = 2$ Maxwell-Einstein supergravity and Jordan algebras*, *Nucl. Phys.* **B 242** (1984) 244; *Exceptional supergravity theories and the magic square*, *Phys. Lett.* **B 133** (1983) 72.

[8] E. Cartan, *Familles de surfaces isoparamétriques dans les espaces à courbure constante*, *Annali di mat.* **17** (1938) 177; *Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces sphériques*, *Math. Zeitschrift* **45** (1939) 335.

[9] M. Günaydin and M. Zagermann, *The gauging of five-dimensional, $N = 2$ Maxwell-Einstein supergravity theories coupled to tensor multiplets*, *Nucl. Phys.* **B 572** (2000) 131 [hep-th/9912027].

[10] B. de Wit and A. Van Proeyen, *Special geometry, cubic polynomials and homogeneous quaternionic spaces*, *Commun. Math. Phys.* **149** (1992) 307 [hep-th/9112027].

[11] M. Günaydin, G. Sierra and P.K. Townsend, *More on $D = 5$ Maxwell-Einstein supergravity: symmetric spaces and kinks*, *Class. and Quant. Grav.* **3** (1986) 763.

[12] H. Freudenthal, *Beziehungen der E_7 und E_8 zur Oktavenebene: VII*, *Nederl. Akad. Wetensch. Proc.* **A62** (1959) 447;
B.A. Rozenfeld, *Sur certaines classes d'espaces homogènes de groupes de Lie*, *Dokl. Akad. Nauk. SSSR* **106** (1956) 600;

J. Tits, *Interprétation géométriques de groupes de Lie simples compacts de la classe E*, *Mem. Acad. Roy. Belg. Sci.* **29** (1955) 3.

[13] B. de Wit and A. Van Proeyen, *Broken sigma model isometries in very special geometry*, *Phys. Lett.* **B 293** (1992) 94 [hep-th/9207091].

[14] I. Antoniadis, S. Ferrara and T.R. Taylor, *$N = 2$ heterotic superstring and its dual theory in five dimensions*, *Nucl. Phys.* **B 460** (1996) 489 [hep-th/9511108].

[15] T. Mohaupt and M. Zagermann, *Gauged supergravity and singular Calabi-Yau manifolds*, *J. High Energy Phys.* **12** (2001) 026 [hep-th/0109055].

[16] K. Behrndt and G. Dall'Agata, *Vacua of $N = 2$ gauged supergravity derived from non-homogenous quaternionic spaces*, *Nucl. Phys.* **B 627** (2002) 357 [hep-th/0112136].

[17] J.R. Ellis, M. Günaydin and M. Zagermann, *Options for gauge groups in five-dimensional supergravity*, *J. High Energy Phys.* **11** (2001) 024 [hep-th/0108094].

[18] G.A. Diamandis, B.C. Georgalas, P. Kouroumalou and A.B. Lahanas, *On the stability of the classical vacua in a minimal* SU(5) *5-d supergravity model*, *New J. Phys.* **4** (2002) 1 [hep-th/0111046].

[19] P. Jordan, J. von Neumann and E. Wigner, *On an algebraic generalization of the quantum mechanical formalism*, *Ann. Math.* **36** (1934) 29.

[20] See N. Jacobson, *Structure and representations of Jordan algebras*, *Amer. Math. Soc. Coll. Pub.* vol. 39 Rhode Island 1968 and the references therein.

[21] K. Mc Crimmon, *Norms and noncommutative Jordan algebras*, *Pacific J. Math.* **15** (1965) 925.

[22] M. Koecher, *Positivitätsbereiche im $\mathbb{R}^n$*, *Am. J. Math.* **79** (1957) 575;
O.R. Rothaus, *Domains of positivity*, *Abh. Math. Sem. der Univ. Hamburg* **24** (1960) 189;
C. Hertneck, *Positivitätsbereiche und Jordan strukturen*, *Math. Ann.* **146** (1962) 433.

[23] L. Michel and L.A. Radicati, *Properties of the beaking of hadronic internal symmetry*, *Ann. Phys. (NY)* **66** (1971) 758; *Geometry of the octet*, *Ann. Inst. Henri Poincaré* **18** (1973) 185.

[24] E.G. Goodaire, *The derivations of M_n^C* , *Comm. in Algebra* **3** (1975) 21.

[25] B.N. Allison and J.R. Faulkner, *A Cayley-Dickson process for a class of structurable algebras*, *Trans. Amer. Math. Soc.* **283** (1984) 185.

[26] P.K. Townsend, K. Pilch and P. van Nieuwenhuizen, *Selfduality in odd dimensions*, *Phys. Lett.* **B 136** (1984) 38.

[27] E. Bergshoeff et al., *Superconformal $N = 2$, $D = 5$ matter with and without actions*, *J. High Energy Phys.* **10** (2002) 045 [hep-th/0205230].

[28] J. Strathdee, *Extended Poincaré supersymmetry*, *Int. J. Mod. Phys.* **A 2** (1987) 273.

[29] M. Zagermann, *The gauging of vector- and tensor-field-coupled five-dimensional $N = 2$ supergravity*, *Class. and Quant. Grav.* **18** (2001) 3197.

[30] W.-T. Hsiang and W.-Y. Hsiang, *Examples of codimension-one closed minimal submanifolds in some symmetric spaces*, *J. Diff. Geom.* **15** (1980) 543.

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[31] M. Awada and P.K. Townsend, *N=4 Maxwell-Einstein supergravity in five-dimensions and its* SU(2) *gauging*, *Nucl. Phys.* **B 255** (1985) 617;
L.J. Romans, *Gauged $N = 4$ supergravities in five-dimensions and their magnetovac backgrounds*, *Nucl. Phys.* **B 267** (1986) 433;
G. Dall'Agata, C. Herrmann and M. Zagermann, *General matter coupled $N = 4$ gauged supergravity in five dimensions*, *Nucl. Phys.* **B 612** (2001) 123 [hep-th/0103106].

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